A Kochen-Specker system has at least 21 vertices

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A Kochen-Specker system $S$ is a finite set of points on the (open) northern hemisphere, such that there is no 010-coloring; that is: there is no $\{0, 1\}$-valued coloring with

1. three pairwise orthogonal points are assigned $(1, 0, 0), (0, 1, 0)$ or $(0, 0, 1)$ and

2. two orthogonal points are not assigned $(1, 1)$.

point $\sim$ direction of magnetic field in measurement of SPIN-1 coloring $\sim$ non-contextual deterministic theory
A **Kochen-Specker system** $S$ is a finite set of points on the (open) northern hemisphere, such that there is no 010-coloring; that is: there is no $\{0, 1\}$-valued coloring with

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point $\sim$ direction of magnetic field in measurement of SPIN-1 coloring $\sim$ non-contextual deterministic theory

**Theorem (Kochen-Specker)**

*There is a Kochen-Specker system. Thus: there is no non-contextual deterministic theory predicting the measurement of a SPIN-1 particle.*
The smallest Kochen-Specker system?

<table>
<thead>
<tr>
<th>System</th>
<th>Year</th>
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<tbody>
<tr>
<td>Kochen-Specker</td>
<td>1975</td>
<td>\leq 117</td>
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## The smallest Kochen-Specker system?

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Conway’s record
It is a problem about graphs

Given a finite set of points $S$ on the projective plane, its orthogonality graph $\mathcal{G}(S)$ has as vertices the points and two points are adjacent if and only if they are orthogonal.
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A graph $G$ is embeddable if there is a $S$ such that $G \leq G(S)$.
Given a finite set of points $S$ on the projective plane, its orthogonality graph $\mathcal{G}(S)$ has as vertices the points and two points are adjacent if and only if they are orthogonal.

A graph $G$ is embeddable if there is a $S$ such that $G \leq \mathcal{G}(S)$.

A 010-coloring of a graph, is a $\{0,1\}$-vertex coloring, such that

1. every triangle is colored $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$ and
2. no adjacent vertices are colored both 1.
It is a problem about graphs

There is a Kochen-Specker system with $n$ points if and only if there is a embeddable and non-010-colorable graph on $n$ vertices.
Restrict the search

(The orthogonality graph of) a minimal Kochen-Specker system is connected; $\sim 10^{26.4}$. 


(The orthogonality graph of) a minimal Kochen-Specker system is connected; squarefree and

\[ \sim 10^{26.4} \]
\[ \sim 10^{10.2} \]
Restrict the search

(The orthogonality graph of) a minimal Kochen-Specker system is connected; \(\sim 10^{26.4}\) squarefree and \(\sim 10^{10.2}\) has minimal vertex degree 3; \(\sim 10^{7.5}\)
The candidates

Our computation found the following number of non 010-colorable squarefree graphs with minimal vertex degree 3

<table>
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<th>$# V$</th>
<th>$#$ candidates</th>
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<td>$\leq 16$</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>2</td>
</tr>
<tr>
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Unembeddable subgraphs

All these candidates contain as a subgraph one of these unembeddable graphs.
Suppose this graph is embeddable.

Note that $v$ and $a$ are distinct points orthogonal to $p_1$. Thus $p_1$ is fixed. Observe: $p_1$ is collinear to $v \times a$. 
Suppose this graph is embeddable.

Note that \( v \) and \( a \) are distinct points orthogonal to \( p_1 \). Thus \( p_1 \) is fixed. Observe: \( p_1 \) is collinear to \( v \times a \).

Similarly: \( p_2 \) is collinear to \( v \times (v \times a) \). And so on. We see \( a \) is collinear to \( x \times (x \times (w \times (w \times (v \times (v \times a)))))) \).

Pen and paper proof of unembeddability
Pen and paper proof of unembeddability

We may assume $z = (0, 0, 1)$, $x = (1, 0, 0)$, $v = (v_1, v_2, 0)$, $w = (w_1, w_2, 0)$ and $a = (0, a_2, a_3)$. We have:

$$\begin{pmatrix} 0 \\ a_2 \\ a_3 \end{pmatrix}$$ is collinear to $$\begin{pmatrix} 0 \\ -a_2 v_1 w_2 (v_1 w_1 + v_2 w_2) \\ -a_3 (v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2) \end{pmatrix}$$

Since $v$ and $w$ are not collinear, we have by Cauchy-Schwarz $|\langle v, w \rangle| < 1$. Note $|v_1|, |w_2| \leq 1$. Thus:

$$|v_1 w_2 \langle v, w \rangle| < 1.$$
Pen and paper proof of unembeddability

We may assume $z = (0, 0, 1)$, $x = (1, 0, 0)$, $\nu = (v_1, v_2, 0)$, $w = (w_1, w_2, 0)$ and $a = (0, a_2, a_3)$. We have:

$$\begin{pmatrix} 0 \\ a_2 \\ a_3 \end{pmatrix} \text{ is collinear to } \begin{pmatrix} 0 \\ -a_2 v_1 w_2 (v_1 w_1 + v_2 w_2) \\ -a_3 (v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2) \end{pmatrix}$$

That is:

$$v_1 w_2 \langle \nu, w \rangle = v_1 w_2 (v_1 w_1 + v_2 w_2)$$
$$= v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$$
$$= (v_1^2 + v_2^2) w_1^2 + (v_1^2 + v_2^2) w_2^2$$
$$= w_1^2 + w_2^2 = 1.$$
Pen and paper proof of unembeddability

We may assume \( z = (0, 0, 1) \), \( x = (1, 0, 0) \), \( v = (v_1, v_2, 0) \), \( w = (w_1, w_2, 0) \) and \( a = (0, a_2, a_3) \). We have:

\[
\begin{pmatrix} 0 \\ a_2 \\ a_3 \end{pmatrix} \text{ is collinear to } \begin{pmatrix} 0 \\ -a_2 v_1 w_2 (v_1 w_1 + v_2 w_2) \\ -a_3 (v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2) \end{pmatrix}
\]

That is:

\[
v_1 w_2 \langle v, w \rangle = v_1 w_2 (v_1 w_1 + v_2 w_2) = v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2 = (v_1^2 + v_2^2) w_1^2 + (v_1^2 + v_2^2) w_2^2 = w_1^2 + w_2^2 = 1.
\]

Since \( v \) and \( w \) are not collinear, we have by Cauchy-Schwarz

\(| \langle v, w \rangle | < 1\). Note \(|v_1|, |w_2| \leq 1\). Thus: \(|v_1 w_2 \langle v, w \rangle | < 1\).

Contradiction.
Example of automized cross product chasing

load_package redlog;
rlset R;
procedure d(x,y);
    (first x) * (first y) +
    (second x) * (second y) +
    (third x) * (third y);
procedure k(x,y);
    { (second x)*(third y) - (third x)*(second y),
      (third x)*(first y) - (first x)*(third y),
      (first x)*(second y) - (second x)*(first y) };
v0c1 := 1; v0c2 := 0; v0c3 := 0;
v1c1 := 0; v1c2 := 1; v1c3 := 0;
v0 := {v0c1, v0c2, v0c3};
v1 := {v1c1, v1c2, v1c3};
v2 := {v2c1, v2c2, v2c3};
v3 := {v3c1, v3c2, v3c3};
v2c1 := 0;
neq0 := k(v0,k(v3,v1));

(snip)

neq29 := k(k(k(k(v3,v1),v1),v2),k(k(v3,v0),v3));
phi :=
    (first neq0 neq 0 or
    second neq0 neq 0 or
    third neq0 neq 0) and

(snip)

    (first neq29 neq 0 or
    second neq29 neq 0 or
    third neq29 neq 0) and
d(v2,v0) = 0 and
d(k(k(v3,v0),v3),k(k(k(v3,v1),v1),v2),v2)) = 0 and
true;
rlqe ex(v3c3,
ex(v3c2,
ex(v3c1,
ex(v2c3,
ex(v2c2,phi))))));
Source code, paper and experimental results can be found at 

kochen-specker.info
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Some open problems:

- If $G$ is embeddable, is there a $S$ such that $G = G(S)$.
- Is every embeddable graph, grid embeddable? That is: using points of the form $(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}}, \frac{z}{\sqrt{n}})$ for $x, y, z, n \in \mathbb{Z}$. 